(ES) Calcolare il flusso TOT. ~~[illegible]~~ del campo vettoriale $\vec{F}$ definito da

12ª lez
23/05/17
(E11)

$$\vec{F}(x,y) = \left(3xy^2 \sin x \,,\ -xy^3 \cos x + x^2 y\right)$$

attraverso la frontiera del dominio:

$$D = \left\{(x,y) \in \mathbb{R}^2 \mid -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2},\ 0 \le |y| \le \cos x \right\}$$

Teorema della Div.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}\left(3xy^2 \sin x\right) + \frac{\partial}{\partial y}\left(-xy^3 \cos x + x^2 y\right)$$

$$= 3y^2\left[\sin x + x \cos x\right] +$$

$$+ x\left[-3y^2 \cos x + x\right] = 3y^2 \sin x + \cancel{3y^2 x \cos x} +$$

$$- \cancel{3y^2 x \cos x} + x^2 = 3y^2 \sin x + x^2$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \int_{-\cos x}^{\cos x} \left(3y^2 \sin x + x^2\right) dy\, dx =$$

$$= 2 \int_{-\pi/2}^{\pi/2} \left[3 \sin x \int_0^{\cos x} y^2\, dy + x^2 \int_0^{\cos x} dy\right] dx$$

$$= 2\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{sen} x \cos^3 x \, dx}_{=0} + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x \, dx = \qquad \text{E12}$$

= 0 (funz. integranda dispari e intervallo simm. risp. all'origine)

$$= 4\int_0^{\frac{\pi}{2}} x^2 \cos x \, dx \overset{\text{x parti}}{=} \dots = \pi^2 - 8.$$

□

Formula di Stokes

$D \subset \mathbb{R}^2$ dominio regolare, $\vec{F} = (X, Y)$ con $X, Y \in C^1(D)$.

Allora:

$$\int_{+\partial D} X\,dx + Y\,dy = \iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\,dy$$

Dim

Applichiamo le formule di Gauss-Green:

$$\iint_D \frac{\partial Y}{\partial x}\,dx\,dy = \int_{+\partial D} Y\,dy \qquad \ominus \Rightarrow \dots \text{ tesi.} \quad \square$$

$$\iint_D \frac{\partial X}{\partial y}\,dx\,dy = -\int_{+\partial D} X\,dx.$$

ESERCITAZIONI

ESAME 21/06/2016

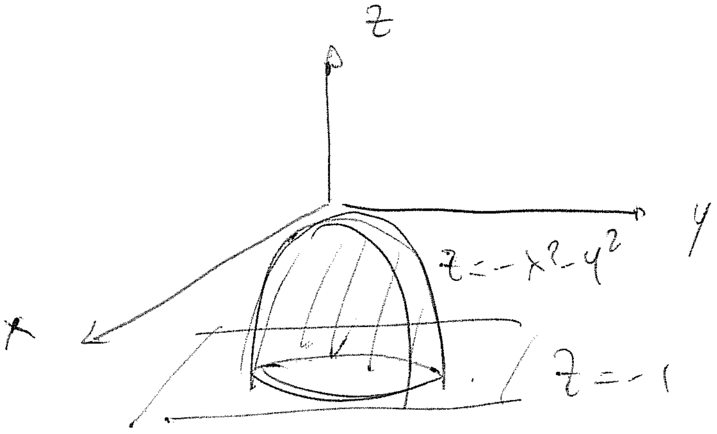
F1

(1) Calcolare

$$\int_{+\partial V} \vec{F} \cdot \vec{n} \, d\sigma \qquad \text{con } \vec{F} = (x, y, z^2)$$

V solido delimitato dalle superfici

$z = -x^2 - y^2$ e dal piano $z = -1$



TEOR. DIV

$\underline{-1 \le z \le -x^2 - y^2}$

CONDIZ. DI COMPATIB.

$-1 \le -x^2 - y^2$

$\Longleftrightarrow \boxed{x^2 + y^2 \le 1}$

$\boxed{F_2}$

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases} \qquad \begin{array}{l} \vartheta \in [0, 2\pi] \\ \rho \in [0, 1] \end{array}$$

$$\Rightarrow \quad \underbrace{-1 \le z \le -\rho^2}$$

$$\oint_{\partial D} \int_{+\partial V} \vec{F} \cdot \vec{n} \, d\sigma = \iiint_V \operatorname{div} \vec{F} \, dx\,dy\,dz$$

$\Uparrow$

integ. superficiale

$$\operatorname{div} \vec{F} = 1 + 1 + 2z = 2(1+z)$$

$$\Rightarrow \quad 2 \int_0^{2\pi} \int_0^1 \int_{-1}^{-\rho^2} (1+z)\,\rho \, dz\, d\rho\, d\vartheta$$

$$= \cancel{4}^{2}\pi \int_0^1 \rho \left. \frac{[1+z]^2}{\cancel{2}} \right|_{-1}^{-\rho^2} d\rho$$

$$= 2\pi \int_0^1 \rho \left\{ (1-\rho^2)^2 - 0 \right\} d\rho$$

T3

$$= -\frac{2\pi}{2} \int_0^1 (-2\rho)\,(1-\rho^2)^2 \, d\rho$$

$$= -\pi \left. \frac{(1-\rho^2)^3}{3} \right|_0^1 = \pi/3 \quad \boxtimes$$

<u>NB</u>

$$\vec{n} = \frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|}, \qquad d\sigma = |\vec{r_u} \times \vec{r_v}| \, du\, dv$$

$\vec{r}$: parametrizz. della superf. $\boxtimes$

(2) Stabilire in quali regioni del piano la seguente forma diff. è esatta

$$\omega = \frac{2(y-x)}{1-(y-x)^2}\, dx + \frac{2(x-y)}{1-(y-x)^2}\, dy$$

Calcolare l'integrale della forma diff. lungo la curva $\gamma(t)$ seguente

$$\gamma(t) = \left(t, \ \frac{\sin \pi t}{2 + \cos t} + \frac{3}{2} + t \right), \quad t \in [0, 1]$$

$$X = 2\,\frac{(y-x)}{1-(y-x)^2} \Rightarrow \cancel{X_y = 2\,\frac{1-(y-x)^2+(y-x)\,2\,(y-x)}{\left[1-(y-x)^2\right]^2}}$$

$$Y = 2\,\frac{(x-y)}{1-(y-x)^2} \Rightarrow \cancel{Y_x = 2\,\frac{1-(y-x)^2+(x-y)\,2\,(y-x)}{\left[1-(y-x)^2\right]^2}}$$

$$X_y = 2\,\frac{1-(y-x)^2 - (y-x)\cdot\left[-2(y-x)\right]}{\left[1-(y-x)^2\right]^2}$$

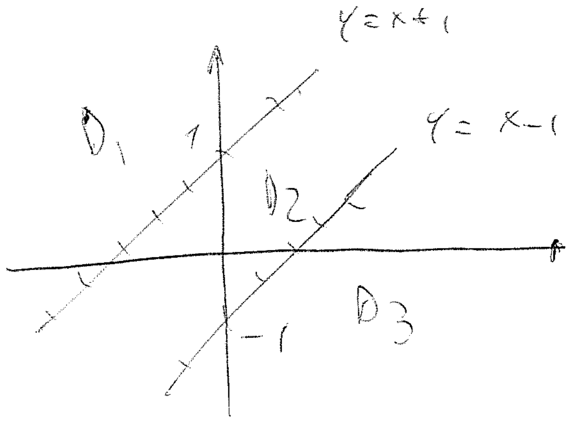
$$Y_x = 2\,\frac{1-(y-x)^2 - (x-y)\left[-2(y-x)\right](-1)}{\left[\quad\right]^2}$$

$$\Rightarrow \boxed{X_y = Y_x}$$

Inoltre $1-(y-x)^2 \neq 0 \Rightarrow y-x \neq \pm 1$

$$y \neq 1 + x$$

$$y \neq -1 + x$$



D_1, D_2, D_3 sempl. connessi

$\Rightarrow$ Per il Teorema 4 la FORMA ω è esatta in D_1, D_2, D_3. (Lez. 09/05)

~~Calcoliamo~~ Posto $y = \underbrace{\dfrac{\sin t}{2+\cos t}}_{\geq 0} + \dfrac{3}{2} + t$

$\Rightarrow y \geq \dfrac{3}{2} + t > 1 + t = 1 + x$

Quindi γ si trova in D_1

Possiamo allora applicare il TEOREMA 1 (Lez 02/05) perché abbiamo ω ESATTA, D_1 sempl. connesso, γ in D_1:

$$\int_{\gamma(P', P'')} X\,dx + Y\,dy = f(P'') - f(P')$$

f è la primitiva della FORMA

F6

Cerchiamo f t.c. $df = f_x\, dx + f_y\, dy$

$= X\, dx + Y\, dy$

$\Rightarrow f_x = X \Rightarrow f(x,y) = \int X\, dx + g(y)$

$f(x,y) = \int \frac{2(y-x)}{1-(y-x)^2}\, dx + g(y)$ in D_1

$= \ln|1-(y-x)^2| + g(y)$ in D_1

~~Imponiamo $f_y = Y \Rightarrow$~~ $= \ln[(y-x)^2 - 1] + g(y)$ in D_1

Imponiamo $f_y = Y \Rightarrow$

$\Rightarrow \frac{2(y-x)}{(y-x)^2 - 1} + g'(y) = \frac{2(x-y)}{1-(y-x)^2}$

$= \frac{2(y-x)}{(y-x)^2 - 1}$

$\Rightarrow \boxed{g'(y) = 0} \Rightarrow g(y) = \text{cost}$

F7

$$f(x,y) = \ln\left[(y-x)^2 - 1\right] + C$$

$$t = 0 \rightarrow \gamma(0) = \left(0, \frac{3}{2}\right) = P'$$

$$t = 1 \rightarrow \gamma(1) = \left(1, \frac{5}{2}\right) = P''$$

$$\Rightarrow \int_\gamma \omega = f\left(1, \frac{5}{2}\right) - f\left(0, \frac{3}{2}\right) =$$

$$= \ln\frac{5}{4} - \ln\frac{5}{4} = 0$$

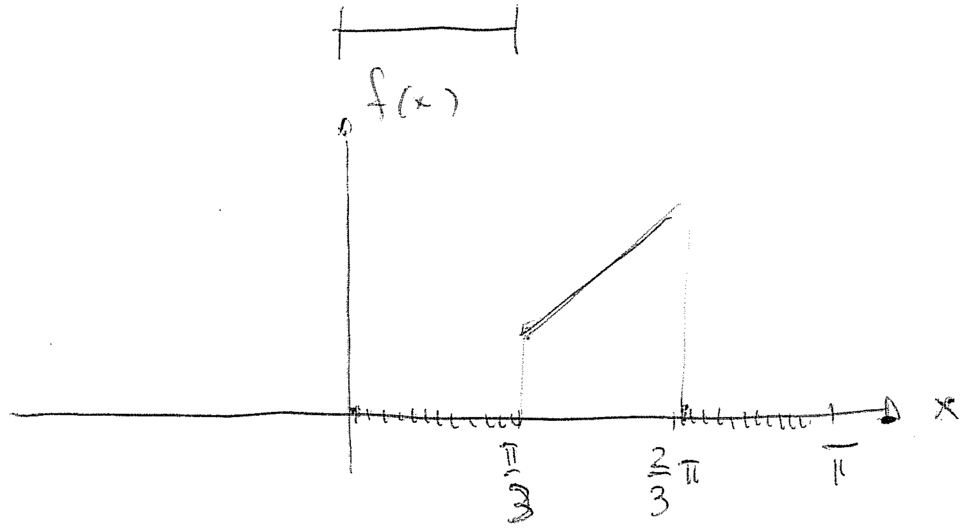
3) Sviluppare in serie di Fourier di soli seni nell'intervallo $[0, \pi]$ la funzione

$$f(x) = \begin{cases} 0 & 0 \le x < \pi/3 \\ 1 + x/2 & \frac{\pi}{3} \le x \le \frac{2}{3}\pi \\ 0 & \frac{2}{3}\pi < x \le \pi \end{cases}$$

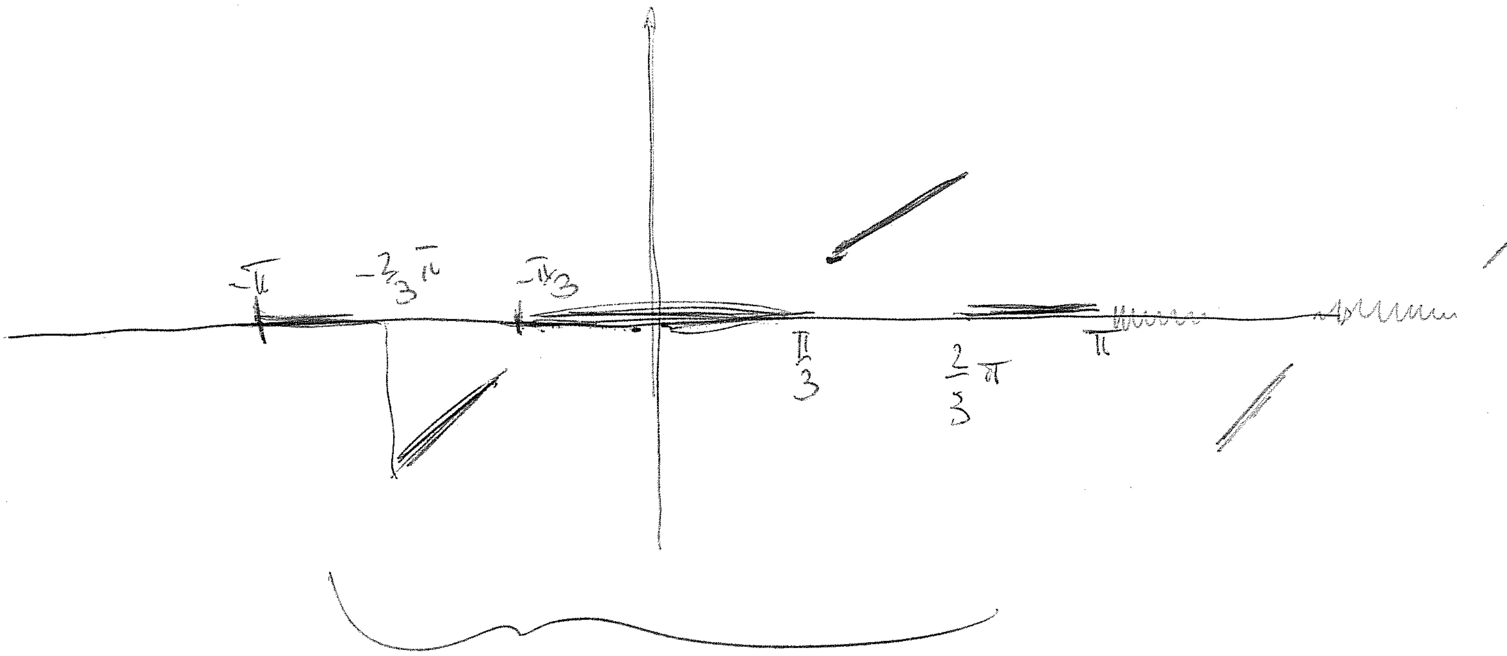
esaminando la convergenza della serie ottenuta.

Disegno dell'estensione periodica corrispondente

F8



estensione periodica:



2π (PROLUNG. DISPARI)

$$\sum_{k=1}^{\infty} b_k \sin kx \quad , \quad b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx \, dx$$

$$b_k = \frac{2}{\pi}\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \left(1+\frac{x}{2}\right)\sin kx \, dx$$

(f9)

$g' = \sin kx \Rightarrow g = \int \sin kx \, dx$

$= -\frac{1}{k} \cos kx$

$\boxed{\text{PER PARTI}}$

$f = 1 + \frac{x}{2} \Rightarrow f' = \frac{1}{2}$

$$b_k = \frac{2}{\pi} \left[-\left(1+\frac{x}{2}\right)\frac{1}{k} \cos kx + \frac{1}{2k} \int \cos kx \, dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{1}{k}\left(1+\frac{x}{2}\right) \cos kx + \frac{\sin kx}{2k^2} \right]_{\frac{\pi}{3}}^{\frac{2}{3}\pi}$$

$$= \frac{2}{\pi} \left[-\frac{1}{k}\left(1+\frac{\pi}{3}\right) \cos\left(\frac{2k\pi}{3}\right) + \frac{\sin 2k\pi}{2k^2} + \right.$$

$$\left. + \frac{1}{k}\left(1+\frac{\pi}{6}\right) \cos \frac{k\pi}{3} - \frac{\sin k\pi/3}{2k^2} \right]$$

NB

$$\cancel{\sin(k\pi - x) =}$$

$$\cancel{\cos(k\pi - x) =}$$

- $$\cos\left(\frac{2k\pi}{3}\right) = \cos\left(k\pi - \frac{k\pi}{3}\right) = \cos k\pi \, \cos \frac{k\pi}{3} = (-1)^k \cos\left(\frac{k\pi}{3}\right)$$

- $$\sin\left(\frac{2k\pi}{3}\right) = \sin\left(k\pi - \frac{k\pi}{3}\right) = -\cos k\pi \, \sin \frac{k\pi}{3} = (-1)^{k+1} \sin \frac{k\pi}{3}$$

$\cancel{\Rightarrow b_k =}$ $\quad b_k = \ldots\ldots$ PARI

DISPARI

$$\Rightarrow \sum_k b_k \sin kx = \begin{cases} f(x) & x \neq \frac{k\pi}{3}, \frac{2k\pi}{3} \\ \frac{1}{2} + \frac{\pi}{12} & x = \frac{k\pi}{3} \\ \frac{1}{2} + \pi/6 & x = \frac{2k\pi}{3} \end{cases}$$

...

OSS

- La serie $\sum_k f_k(x)$ conv. tot. in I

se converge la serie numerica:

$$\sum_k \sup_{x\in I} \| f_k(x) \|$$

MODULO

- La serie $\sum_{k=1}^{\infty} f_k(x)$ conv. unif. se converge

unif. la succ. delle somme parziali:

$$\left\{ \sum_{k=1}^{n} f_k(x) \right\} = \left\{ S_n(x) \right\}$$

~~cioè~~ cioè se $\sup_{x\in I} |S_n(x) - S(x)| \xrightarrow{n\to+\infty} 0$

$\lim_{n\to+\infty} S_n(x) = S(x)$

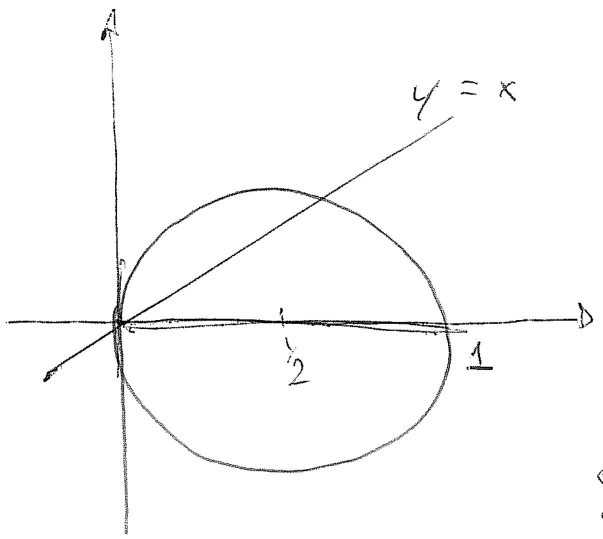
ESAME 13/06/2015 F11

(1) Calcolare

$$\iint_D \sqrt{1-x^2-y^2}\;dx\,dy$$

dove D è il DOMINIO PIANO delimitato dalle curve $x^2+y^2=x$ e dalle semirette $y=x,\ x\geq 0$ e

$y=0 \quad x\geq 0$.



$$x^2-x+y^2=0$$

$$x^2-x+\tfrac{1}{4}+y^2=\tfrac{1}{4}$$

$$\left(x-\tfrac{1}{2}\right)^2+y^2=\tfrac{1}{4}$$

Se $\begin{cases} x=\rho\cos\vartheta \\ y=\rho\cos\vartheta \end{cases} \Rightarrow \vartheta\in\left[0,\frac{\pi}{4}\right]$ $(0\leq y\leq x)$

Ma $x^2+y^2\leq x \Rightarrow \rho^2\leq\rho\cos\vartheta$

PUNTI INTERNI

$\Rightarrow \boxed{\rho\leq\cos\vartheta}$

$$\boxed{T12}$$

$$\Rightarrow I = \int_0^{\pi/4} \int_0^{\cos\vartheta} \sqrt{1-\rho^2}\ \rho\, d\rho\, d\vartheta$$

$$= \int_0^{\pi/4} -\frac{1}{2} \left[(1-\rho^2)^{3/2} \cdot \frac{2}{3} \right]_0^{\cos\vartheta} d\vartheta$$

$$= -\frac{1}{3} \int_0^{\pi/4} \left[(1-\cos^2\vartheta)^{3/2} - 1 \right] d\vartheta$$

$$= -\frac{1}{3} \int_0^{\pi/4} (\sin^3\vartheta - 1)\, d\vartheta =$$

$$= -\frac{1}{3} \int_0^{\pi/4} \sin\vartheta \sin^2\vartheta\, d\vartheta + \frac{\pi}{12} =$$

$$= -\frac{1}{3} \int_0^{\pi/4} \sin\vartheta\, (1-\cos^2\vartheta)\, d\vartheta + \frac{\pi}{12}$$

$$= \frac{1}{3} \cos\vartheta \Big|_0^{\pi/4} - \frac{1}{9} \cos^3\vartheta \Big|_0^{\pi/4} + \frac{\pi}{12}$$

$$= \frac{1}{3} \left(\frac{\sqrt{2}}{2} - 1 \right) - \frac{1}{9} \left[\left(\frac{\sqrt{2}}{2} \right)^3 - 1 \right] + \frac{\pi}{12} = \ldots$$

② Calcolare $\quad$ F13

$$\int_{+\partial V} \vec{F} \cdot \vec{n} \, d\sigma \qquad \text{con} \quad \vec{F} = \left(\frac{4}{3} x^3, \, z, \, 4z(x^2+y^2) \right)$$

dove V è il volume delimitato dalle superficie

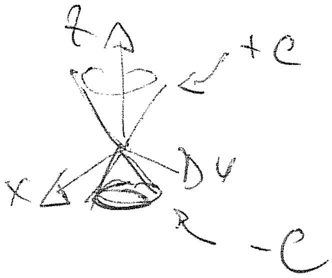
$z + 1 = \sqrt{x^2+y^2}$ e dei piani $z=2$, $z=3$

$\vdash\!\dashv$

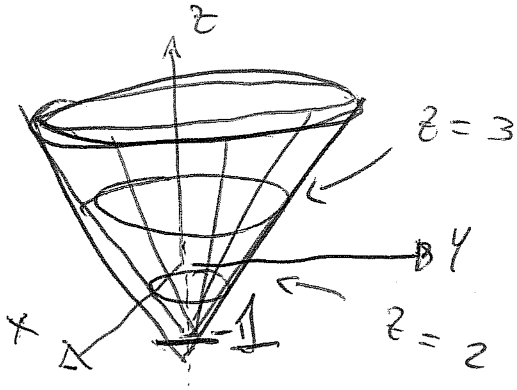
<u>TEOR. DIV.</u>

NB: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$ eq cono.

$$c^2\left[\frac{x^2}{a^2} + \frac{y^2}{b^2}\right] = z^2 \Rightarrow z = \pm c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

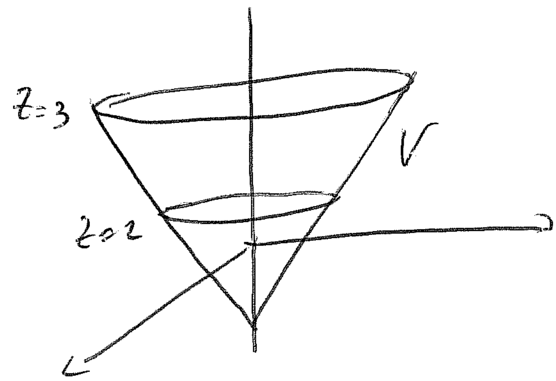


Quindi abbiamo:



(F.15)

$$\Rightarrow \int_{+\partial V} \vec{F} \cdot \vec{n}\, d\sigma = \iiint_V \operatorname{div} \vec{F}\, dx\,dy\,dz$$



$$\operatorname{div} \vec{F} = 4x^2 + 0 + 4(x^2+y^2)$$

Dobbiamo calcolare quindi $\iiint_V \left[4x^2 + 4(x^2+y^2)\right] dx\,dy\,dz$

~~1° MODO~~ ~~1° MODO: INTEG. PER STRATI~~

COORDINATE CILINDRICHE
E INT. PER STRATI

$$V = \{ (x,y,z) : h_1 \le z \le h_2,\ (x,y) \in \Omega(z) \}$$

DOMINIO REG. del piano

Nel nostro caso

$$V = \{ (x,y,z) : z \in [2,3],\ x^2+y^2 \le (z+1)^2 \}$$

$$\begin{cases} x = \rho \cos\vartheta \\ y = \rho \sin\vartheta \end{cases}$$

$\vartheta \in [0, 2\pi]$

$0 \le \rho \le z+1$

F15

$$\Rightarrow 4\int_0^{2\pi}\int_2^3\int_0^{z+1}\left[\rho^2\cos^2\vartheta+\rho^2\right]\rho\,d\rho\,dz\,d\vartheta$$

$$=4\int_0^{2\pi}\int_2^3\int_0^{z+1}\left[\rho^3\cos^2\vartheta+\rho^3\right]d\rho\,dz\,d\vartheta$$

$$=\int_0^{2\pi}\int_2^3\left[\cos^2\vartheta\left(\rho^4\right)_0^{z+1}+\left(\rho^4\right)_0^{z+1}\right]dz\,d\vartheta$$

$$=\int_0^{2\pi}\int_2^3(z+1)^4\cos^2\vartheta\,dz\,d\vartheta+2\pi\int_2^3(z+1)^4\,dz$$

$$=\frac{(z+1)^5}{5}\Bigg|_2^3\cdot\left[\int_0^{2\pi}\frac{1+\cos2\vartheta}{2}\,d\vartheta+2\pi\right]=$$

$$=\frac{3\pi}{5}\left(4^5-3^5\right)\qquad\underline{\underline{ok}}$$

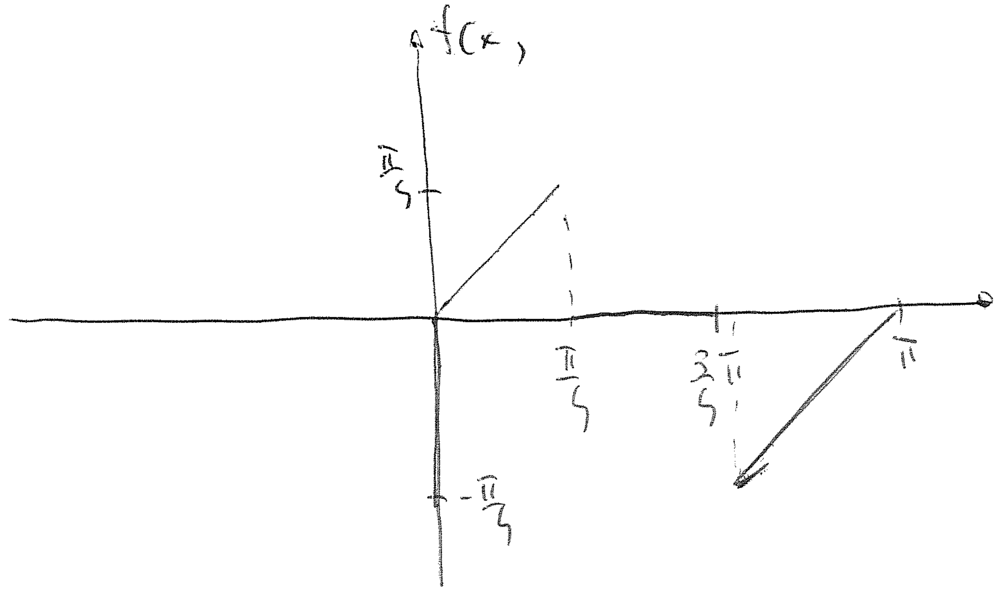
1° MODO: COORDINATE EQUIPARATE

16

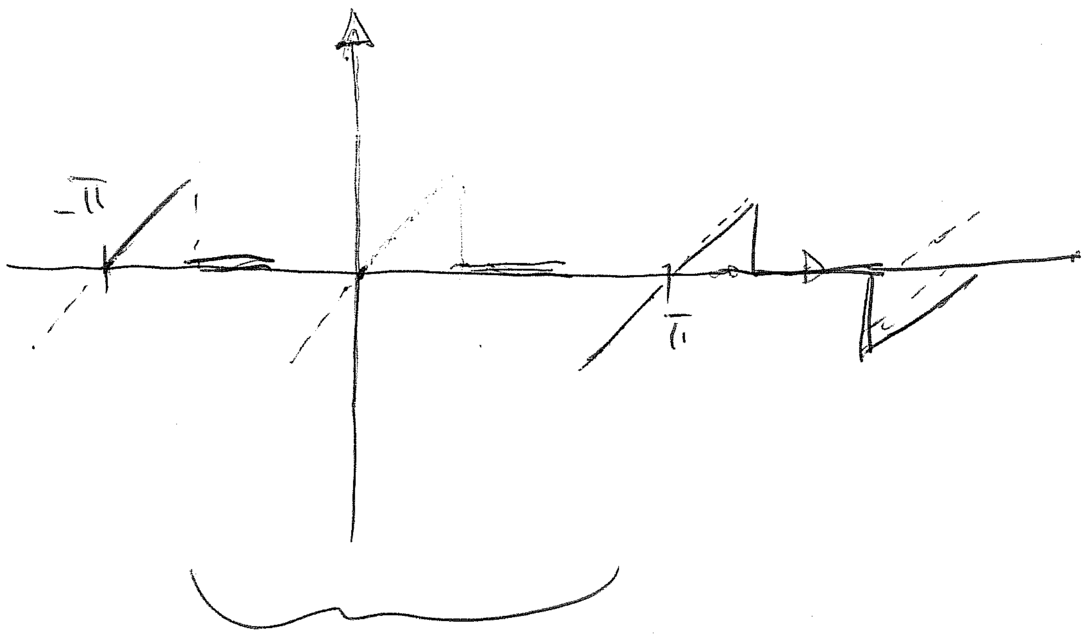
(3)

$$f(x) = \begin{cases} x & 0 \leq x \leq \pi/4 \\ 0 & \pi/4 < x < \frac{3}{4}\pi \\ x - \pi & \frac{3}{4}\pi \leq x \leq \pi \end{cases}$$

Sviluppare in serie di Fourier di soli seni la funz. $f(x)$. Esaminare la conv. della ~~serie~~ serie ottenuta e disegnare il grafico dell'estensione periodica corrisp.



F17



2π Prolungamento dispari (quali sono quelle funz. che hanno uno sviluppo in soli seni?)

FUNZ. DISC. (CONV. PUNTUALE) ⟸

$$\sum_k b_k \sin kx = ?. \qquad b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx\,dx$$

$$b_k = \frac{2}{\pi}\left[\int_0^{\pi/5} x \sin kx\,dx + \int_{\frac{3}{5}\pi}^{\pi}(x-\pi)\sin kx\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{x}{k}\cos kx\Big|_0^{\pi/5} + \frac{1}{k}\int_0^{\pi/5}\cos kx\,dx + \right.$$

$$\left. -(x-\pi)\frac{\cos kx}{k}\Big|_{\frac{3}{5}\pi}^{\pi} + \frac{1}{k}\int_{\frac{3}{5}\pi}^{\pi}\cos kx\,dx\right]$$

$$b_k = \frac{2}{\pi}\left[-\frac{\pi}{4k}\cos\frac{\pi}{4}k + \frac{1}{k^2}\sin k\frac{\pi}{4} + \right.$$

$$\left. -\frac{\pi}{4k}\cos\left(\frac{3}{4}\pi k\right) - \frac{\sin\left(\frac{3}{4}\pi k\right)}{k^2} \right]$$

$$= -\frac{1}{2k}\cos\frac{\pi k}{4} + \frac{2}{\pi k^2}\sin k\frac{\pi}{4} +$$

$$-\frac{1}{2k}\cos\left(\frac{3\pi k}{4}\right) - \frac{2}{\pi k^2}\sin\left(\frac{3\pi k}{4}\right)$$

NB: $\cos\left(\frac{3\pi k}{4}\right) = \cos\left(k\pi - \frac{\pi}{4}k\right) = (-1)^k \cos\frac{k\pi}{4}$

$\sin\left(\frac{3\pi k}{4}\right) = \sin\left(k\pi - \frac{\pi}{4}k\right) = (-1)^{k+1} \sin\frac{k\pi}{4}$

$$b_k = -\frac{1}{2k}\cos\frac{\pi k}{4} + \frac{2}{\pi k^2}\sin k\frac{\pi}{4} +$$

$$+ \frac{(-1)^{k+1}}{2k}\cos\left(\frac{k\pi}{4}\right) + \frac{2}{\pi k^2}(-1)^{k+2}\sin\left(\frac{k\pi}{4}\right)$$

$= \ldots$

7 (9)

$$\sum_k b_k \sin kx = \begin{cases} f(x), & 0 \le x \le \pi, \quad x \neq \frac{\pi}{4}, \frac{3}{4}\pi \\ \frac{\pi}{8}, & x = \frac{\pi}{4} \\ -\frac{\pi}{8}, & x = \frac{3}{4}\pi \end{cases}$$

ESAME 15/07/2015

(1) Si consideri la forma diff:

$$\omega = \frac{2xy}{(1+x^2)^2}\,dx - \frac{1}{1+x^2}\,dy$$

- stabilire se ω è chiusa
- dire se ω è esatta ed in caso affermativo determinare tutte le sue primitive.
- calcolare l'int. curvilineo $\int_\gamma \omega$ dove γ è il perimetro del poligono che ha per vertici i punti:

$O(0;0)$ $A(1,2)$ $B(3,-7)$

$C(-5,-3)$ $D(10,0)$