FORMULE DI GAUSS-GREEN

[11ª lezione]
16/05/17

(E4)

$D \subset \mathbb{R}^2$ DOMINIO REGOLARE

$f = f(x,y) \in C^1(D)$

$$\Rightarrow \quad \iint_D \frac{\partial f}{\partial x}\, dx\, dy = \int_{+\partial D} f \, dy$$

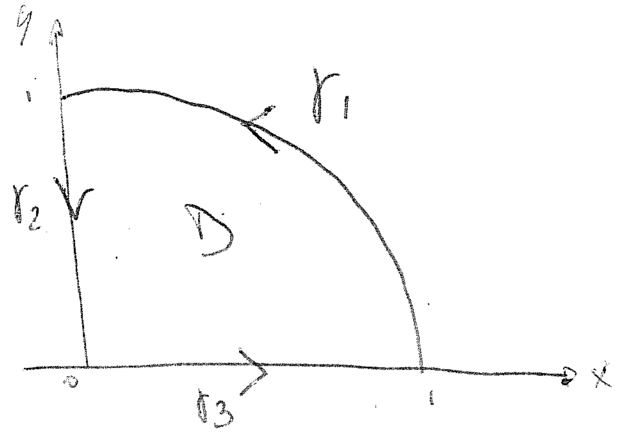
$$\iint_D \frac{\partial f}{\partial y}\, dx\, dy = -\int_{+\partial D} f \, dx$$

Es. $$\iint_D \frac{x^2 y}{\sqrt{x^2+y^2}}\, dx\, dy$$

D è la porzione di cerchio unitario di centro l'origine contenuta nel 1° quadrante:

GAUSS-GREEN

$$\iint_D \frac{\partial}{\partial y}\left[x^2\sqrt{x^2+y^2}\right] dx\, dy$$



$$\Rightarrow \iint_D \frac{\partial}{\partial y}\left(x^2 \sqrt{x^2+y^2} \right) dx\, dy =$$

$$= - \int_{+\partial D} x^2 \sqrt{x^2+y^2}\, dx$$

$$\gamma_1 : \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t \in [0, \pi/2]$$

$$+\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

$$\gamma_2 : \begin{cases} x = 0 \\ y = t \end{cases} \quad t \in [1, 0]$$

$$\gamma_3 : \begin{cases} x = t \\ y = 0 \end{cases} \quad t \in [0, 1]$$

$$\Rightarrow \iint_D \cdots = - \int_{+\partial D} \cdots = - \int_{\gamma_1} x^2 \sqrt{x^2+y^2}\, dx \; +$$

$$- \int_{\gamma_2} x^2 \sqrt{x^2+y^2}\, dx - \int_{\gamma_3} x^3 \sqrt{x^2+y^2}\, dx$$

$$= - \int_0^{\pi/2} \cos^2 t \cdot 1\, (-\sin t)\, dt - \int_{\gamma_2} 0 - \int_0^1 t^2 \sqrt{t^2+0}\, dt$$

$$= \int_0^{\pi/2} \sin t \cos^2 t\, dt - \int_0^1 t^3\, dt = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

COORD. POLARI

$$\begin{cases} x = \rho \cos\vartheta \\ y = \rho \sin\vartheta \end{cases} \qquad \rho \in [0,1], \qquad \vartheta \in [0, \tfrac{\pi}{2}]$$

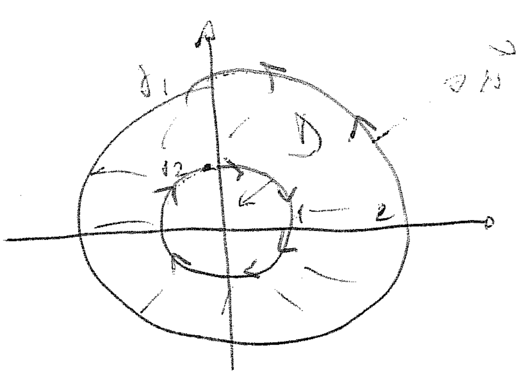
$$\iint_D \frac{x^2 y}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/2}\int_0^1 \frac{\rho^3 \cos^2\vartheta \sin\vartheta}{\cancel{\rho}} \cdot \cancel{\rho}\,d\rho\,d\vartheta$$

$$= \int_0^1 \rho^3\,d\rho \int_0^{\pi/2} \cos^2\vartheta \sin\vartheta\,d\vartheta = \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{12}.$$

M.S.

$$\iint_D x^2(1+y)\,dx\,dy$$

$$D = \{(x,y) \mid 1 \le x^2+y^2 \le 4\}.$$



$$\partial D = \gamma_1 \cup \gamma_2$$

COORD. POLARI

$1 \leq \rho \leq 2 \quad , \qquad \vartheta \in [0, 2\pi]$

$$\int_0^{2\pi} \int_1^2 \rho^3 (1 + \rho \sin\vartheta) \cos^2\vartheta \, d\rho \, d\vartheta$$

$$= \int_0^{2\pi} \int_1^2 \rho^3 \cos^2\vartheta \, d\rho \, d\vartheta + \int_0^{2\pi} \int_1^2 \rho^4 \sin\vartheta \cos^2\vartheta \, d\rho \, d\vartheta$$

$$= \left[\frac{\rho^4}{4}\right]_1^2 \int_0^{2\pi} \frac{1 + \cos 2\vartheta}{2} \, d\vartheta + \left[\frac{\rho^5}{5}\right]_1^2 \cancel{\left[\frac{\cos^3\vartheta}{3}\right]_{2\pi}^{0}} \to 0$$

$$= \left(\frac{16}{4} - \frac{1}{4}\right) \pi = \frac{15}{4} \pi$$

GAUSS-GREEN

$$\iint_D x^2 (1+y) \, dx \, dy = \iint_D \frac{\partial}{\partial x}\left(\frac{x^3}{3}(1+y)\right) dx \, dy$$

$$= \frac{1}{3} \int_{+\partial D} x^3 (1+y) \, dy$$

$$+\partial D = \gamma_1 \cup \gamma_2$$

$$\Rightarrow \frac{1}{3}\int_{\gamma_1} x^3(1+y)\,dy + \frac{1}{3}\int_{\gamma_2} x^3(1+y)\,dy$$

$$\gamma_1: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \qquad t \in [0, 2\pi]$$

$$\gamma_2: \begin{cases} x = \cos t \\ y = \sin t \end{cases} \qquad t \in [2\pi, 0]$$

$$\Rightarrow \frac{1}{3}\int_0^{2\pi} 8\cos^3 t\,(1+2\sin t)\cdot(2\cos t)\,dt +$$

$$+ \frac{1}{3}\int_{2\pi}^{0} \cos^3 t\,(1+\sin t)\cos t\,dt$$

$$= \frac{16}{3}\int_0^{2\pi} \cos^4 t\,(1+2\sin t)\,dt +$$

$$- \frac{1}{3}\int_0^{2\pi} \cos^4 t\,(1+\sin t)\,dt$$

$$= \frac{16}{3} \int_0^{2\pi} \cos^4 t \, dt + \underbrace{\frac{32}{3} \int_0^{2\pi} \sin t \cos^4 t \, dt}_{0} +$$

$$- \frac{1}{3} \int_0^{2\pi} \cos^4 t \, dt - \underbrace{\frac{1}{3} \int_0^{2\pi} \cos^4 t \sin t \, dt}_{0}$$

$$= \frac{15}{3} \int_0^{2\pi} \cos^4 t \, dt = \frac{15}{3} \int_0^{2\pi} (\cos^2 t)^2 \, dt$$

$$= \frac{15}{3} \int_0^{2\pi} \left[\frac{1+\cos 2t}{2}\right]^2 dt = \frac{15}{12} \int_0^{2\pi} (1+\cos 2t)^2 \, dt$$

$$= \frac{15}{12} \int_0^{2\pi} \left[1 + 2\cos 2t + \cos^2 2t\right] dt$$

$$= \frac{15}{12} \int_0^{2\pi} \left[1 + 2\cos 2t + \frac{1+\cos 4t}{2}\right] dt$$

$$= \frac{15}{12} \int_0^{2\pi} \left[\frac{3}{2} + \underbrace{2\cos 2t + \frac{\cos 4t}{2}}_{=0}\right] dt$$

$$= \frac{15}{12} \cdot \frac{3}{2} \cdot \cancel{2}\pi = \frac{15}{4} \pi$$

TEOREMA DELLA DIVERGENZA

D DOMINIO REGOLARE DEL PIANO, $\vec{F} = (X, Y)$

con $X, Y \in C^1(D)$.

Allora:

$$\iint_D \operatorname{div} \vec{F} \, dx\, dy = \int_{+\partial D} \vec{F} \cdot \vec{N} \, ds$$

- $\operatorname{div} \vec{F} \ (= \nabla \cdot \vec{F}) = \dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y}$ DIVERGENZA DEL CAMPO VETT. $\vec{F}$

- $\vec{N}$ normale a ∂D (rivolto verso l'esterno)

- s: ascissa curvilinea su $+\partial D$.

Oss

$\int_{+\partial D} \vec{F} \cdot \vec{N} \, ds$:= FLUSSO TOT del CAMPO $\vec{F}$ USCENTE DA ∂D.

(per una ragione che sarà spiegata dettagliatamente in R^3).

Es) Calcolare il flusso tot. del campo

$\vec{F} = (x, y)$ attraverso la frontiera

del DOMINIO $D = \{(x,y) \mid x^2 + y^2 \le R^2\}$.

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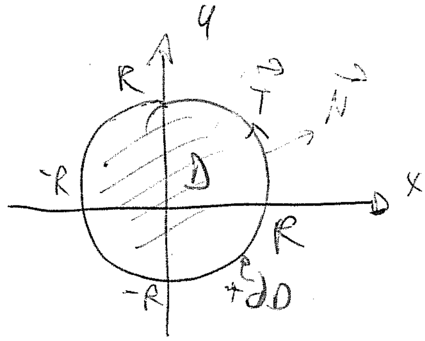
FLUSSO (calcolo diretto)

$$-\int_{+\partial D} \vec{F} \cdot \vec{N} \, ds$$



$$+\partial D: \begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \quad t \in [0, 2\pi]$$

~~Ponendo ... ed~~ Riparametrizzando mediante ascissa curvilinea (si veda es. finale 6ª lezione):

$$+\partial D: \begin{cases} x(s) = R \cos s/R \\ y(s) = R \sin s/R \end{cases} \quad s \in [0, 2\pi R]$$

$$\vec{T}(s) = \vec{r}\,'(s) = \left(-\sin s/R,\ \cos s/R\right)$$

$$\vec{N}(s) = \frac{\vec{T}\,'(s)}{|\vec{T}\,'(s)|} = \frac{\cancel{1/R}\left(-\cos s/R,\ -\sin s/R\right)}{\cancel{1/R}}$$

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$$\vec{F} = (x, y) = \left(R \cos s/R ,\ R \sin s/R \right)$$

$$\Rightarrow \vec{F} \cdot \vec{N} = -R \cos^2 s/R - R \sin^2 s/R = -R$$

$$\Rightarrow -\int_{+\partial D} \vec{F} \cdot \vec{N}\, ds = +R \int_0^{2\pi R} ds = 2\pi R^2$$

T. DIV.

$$\iint_D \underbrace{\operatorname{div} \vec{F}}_{=2}\, dx\, dy = 2 \underbrace{\iint_D dx\, dy}_{\text{area del cerchio: } \pi R^2} = 2\pi R^2$$